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Prediction of Solutions of the Duffing System with Tuned Mass Damper

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In this work we analyze the behavior of a nonlinear dynamical system using a probabilistic approach. We focus on the coexistence of solutions and we check how the changes in the parameters of excitation influence the dynamics of the system. For the demonstration we use the Duffing oscillator with the tuned mass absorber. We mention the numerous attractors present in such a system and describe how they were found with the method based on the basin stability concept.

Keywords: Duffing oscillator tuned mass absorber coexistence of attractors bifurcation analysis attractors detection basin stability.

1. Introduction

The engineers aim to design the robust and predictable systems. At the same time they constantly struggle to increase their performance, which leads to the more complex designs. Some of them are strongly nonlinear and this fact makes them difficult to analyze. In particular, there are the nonlinear systems that are multistable, which means that they can exhibit the qualitatively different behaviors depending on the working conditions [1]. For example the Duffing oscillator, that is used to model many real mechanisms, can be multistable in certain conditions, as it is shown numerically in the aforementioned article. The multistability is not only a matter of the unrealistic computations. A recent research experimentally proved its existence in the system composed of two pendulums [3]. In fact, the phenomenon is widely present in nature, which makes it interesting not only for the mechanical engineers [6,7,9]. Sometimes we encounter a special case of the multistability, when some of the coexisting attractors are very unlikely to occur [5]. Such attractors have the relatively small basins of attraction and therefore they are hard to find in the phase space. The issue becomes even worse when we consider the uncertainty of the parameters describing the system.

Generally, there are no means to solve the complex dynamical systems analytically so we reach for the numerical methods. It turns out that some of the methods, such as the bifurcational analysis based on the path following algorithm, are likely to overlook the non-dominant attractors [2, 4]. Even though the small basins of attraction make these solutions hard to observe in practice, the designer should be aware of their existence. A machine that is not completely predictable would not be acceptable in many applications. In this work we show how to apply the extended basin stability method [3, 8] to track the attractors in the forced duffing oscillator with a tuned mass absorber. The uncertainty of the forcing parameters is taken into account.

2. Model of the System

The analyzed system is shown in Fig. 1. It consists of a Duffing oscillator with a suspended pendulum. The Duffing system is forced by a periodic excitation

$$F(t) = F_0 \cos \nu t.$$

The position of the mass M is given by the coordinate y and the angular displacement of the pendulum (position of the mass m) is given by the angle φ. l is the length of the pendulum, k_1 and k_2 are linear and non–linear parts of spring stiffness and c_1 is a viscous damping coefficient of the Duffing oscillator. We assume that the pendulum is subjected to a small viscous damping c_2 (1% of critical damping), with the damper located in the pivot of the pendulum (not shown in Fig. 1).

One can derive two coupled second order differential equations:

$$(M+m)\ddot{y} - ml\ddot{\varphi}\sin\varphi - ml\dot{\varphi}^2\cos\varphi + k_1 y + k_2 y^3 + c_1\dot{y} = F_0\cos\nu t, \tag{1}$$

$$ml^2\ddot{\varphi} - ml\ddot{y}\sin\varphi + mlg\sin\varphi + c_2\dot{\varphi} = 0. \tag{2}$$

In the numerical calculations we use the following values of parameters:

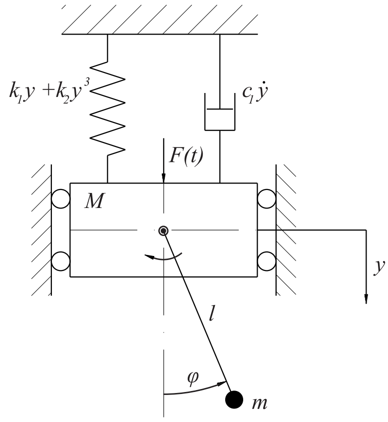

Figure 1 Model of system

$M = 5.0\,[\text{kg}]$, $k_1 = 162.0\left[\frac{\text{N}}{\text{m}}\right]$, $k_2 = 502.0\left[\frac{\text{N}}{\text{m}}\right]$, $c_1 = 3.9\left[\frac{\text{Ns}}{\text{m}}\right]$,
$m = 0.5\,[\text{kg}]$, $l = 0.1\,[\text{m}]$, $c_2 = 0.001\,[\text{Nms}]$.

We neglect the static deflection of mass M.

Introducing dimensionless time $\tau = t\omega_1$, where $\omega_1^2 = \frac{k_1}{M+m}$ is the natural linear frequency of Duffing oscillator, we obtain the dimensionless equations:

$$\begin{aligned} &\ddot{x} - ab\ddot{\gamma}\sin\gamma - ab\dot{\gamma}^2\cos\gamma + x + \alpha x^3 + d_1\dot{x} = f\cos\mu\tau, \\ &\ddot{\gamma} - \tfrac{1}{b}\ddot{x}\sin\gamma + \sin\gamma + d_2\dot{\gamma} = 0, \end{aligned} \tag{3}$$

where

$a = \frac{m}{M+m}$, $b = \left(\frac{\omega_2}{\omega_1}\right)^2$, $\omega_2^2 = \frac{g}{l}$, $\alpha = \frac{k_2 l^2}{(M+m)\omega_1^2}$,
$f = \frac{F_0}{(M+m)l\omega_1^2}$, $d_1 = \frac{c_1}{(M+m)\omega_1}$, $d_2 = \frac{c_2}{ml^2\omega_2}$, $\mu = \frac{\nu}{\omega_1}$,
$x = \frac{y}{l}$, $\dot{x} = \frac{\dot{y}}{\omega_1 l}$, $\ddot{x} = \frac{\ddot{y}}{\omega_1^2 l}$, $\gamma = \varphi$, $\dot{\gamma} = \frac{\dot{\varphi}}{\omega_2}$, $\ddot{\gamma} = \frac{\ddot{\varphi}}{\omega_2^2}$.

The dimensionless parameters of the system have the following values:
$a = 0.091$, $b = 3.33$, $\alpha = 0.031$, $d_1 = 0.132$ and $d_2 = 0.02$.

The dimensionless amplitude f and frequency μ of the excitation are taken as control parameters. Basing on the model from [4] we assume that they can take any values from the ranges $f \in (0, 2.5)$, $\mu \in (0, 3)$ with the equal probability.

3. Sample Based Detection

In this approach we draw the conclusions about the dynamics of the system analyzing its behavior for a finite number of initial conditions and values of control parameters. First, we choose a random point from the accessible phase space, take the random values of the bifurcation parameters from the ranges assumed at the beginning and solve the equations of motion for these values. Then we asses which kind of attractor is represented by the solution obtained. The procedure is repeated multiple times and at the end we try to infer about all the possible solutions basing on the limited number of observations, which is a typical statistical task.

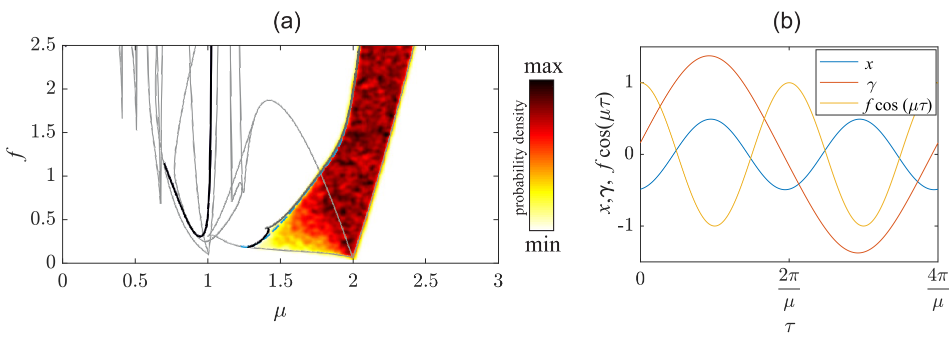

Figure 2 (a) Location of the attractor detected in the parameters plane thanks to the sample based method. The more frequently the attractor appeared in a given region, the darker the region is. (b) Time series characterizing the attractor. Lines plotted for $\mu = 2, f = 1, x_0 = 0, \gamma_0 = 1, \dot{x}_0 = 0, \dot{\gamma}_0 = 0$. The bifurcation lines in the background are taken from [4]

The method is closely related to the concept of basin stability [8]. If a dynamical system is multistable, its phase space is divided into subspaces such that starting from one of the subspaces results in achieving only one kind of attractor. Such a subspace is called the basin of attraction and its size is measured by its volume in the phase space. This means that if a basin of attraction occupies 40% of the phase space and if we randomly choose the points from this phase space, around 40% of the points should belong to this basin and as a result lead to one kind of solution.

Let us take an example from the system presented above. It has four state variables, two bifurcation parameters and therefore six numbers to draw randomly for the simulation. Let us draw 400 000 sets of the six numbers and solve the equations of motion for each set. The number above is an arbitrary large that we judge big enough to well describe the system. In our simulation it turned out that 13% of solutions have common features which indicate that the pendulum swings periodically, performing one movement back and forth per two periods of excitation (Fig. 2(b)). Having the results, we can plot the points with these features on the parameters plane as we show in Fig. 2(a). The plot shows us for which values of parameters the solution can occur and how often. We can see that the shape formed by the points corresponds closely to the bifurcation lines present in the background. The lines were obtained by the path following method and were first presented in [4]. The approach used by Brzeski et al. in this publication was suitable for finding the dominant attractors and the fact that our result is in agreement with theirs is the first indication that the sample based method works correctly.

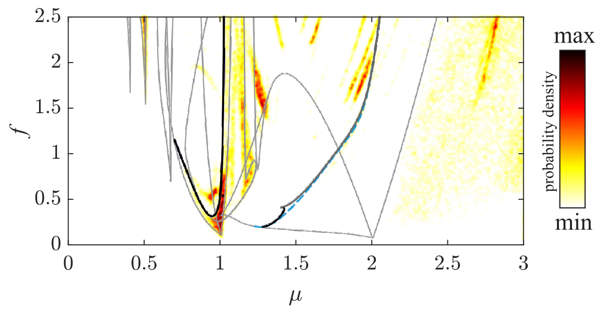

Figure 3 Location of all the rare attractors detected in the parameters plane thanks to the sample based method

We aim here to stress that the method is also suitable for finding the rare attractors hidden in the phase space. Their basins of attraction occupy relatively small volume of the phase space and therefore they are hard to detect. We decided to label an attractor as rare if its probability of occurrence is smaller than 0.5%. Please note that the notion of being rare is relative and depends on the admissible range of parameters and initial conditions. If for example the range of the parameter μ was restricted from $\mu \in (0, 3)$ to $\mu \in (1, 2.5)$, the probability of occurrence of the attractor presented in Fig. 2 would be much higher. Figure 3 shows the distribution of all the rare attractors we were able to detect over the parameters plane. We can see that they are present in many places of the plane and that sometimes they are correlated with the bifurcation lines, but not always.

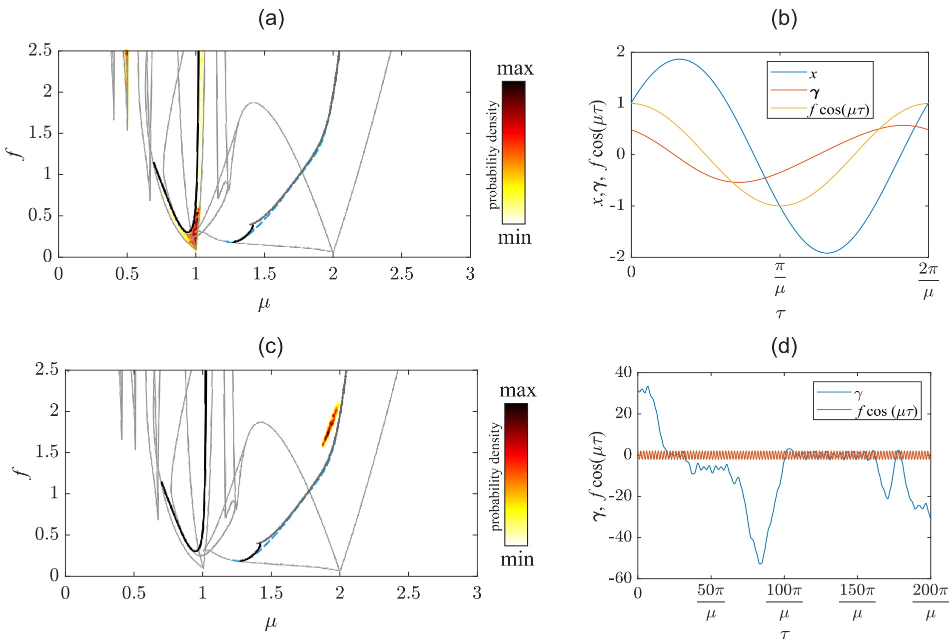

Figure 4 (a) Location of an exemplary rare periodic attractor in the parameters space. (b) Time series characterizing the attractor. One period of excitation corresponds to one period of the pendulum. Lines plotted for $\mu = 1$, $f = 0.3$, $x_0 = -0.9$, $\gamma_0 = 1.6$, $\dot{x}_0 = 1.5$, $\dot{\gamma}_0 = 0.5$. (c) Location of an exemplary rare chaotic attractor in the parameters plane. (d) Time series characterizing the attractor. The pendulum rotates in both directions and occasionally oscillates, which indicates the tumbling chaos. Lines plotted for $\mu = 1.9727$, $f = 2.0275$, $x_0 = 0.2432$, $\gamma_0 = -1.4665$, $\dot{x}_0 = 1.4367$, $\dot{\gamma}_0 = -0.2539$

Periodic attractors			
	Period	**Behavior of pendulum**	**Probability**
1	1	Hangs down	64.21%
2	2	One period of oscillation	13.60%
3	2	One rotation clockwise and one counterclockwise	6.56%
4	1	Three rotations	2.46%
5	1	One rotation	1.32%
6	1	One period of oscillation	0.52%
7	4	One period of oscillation	0.38%
8	2	Six rotations	0.27%
9	8	Four rotations clockwise and four counterclockwise	0.12%
10	10	Ten oscillations of unequal amplitude	0.11%
-	-	All other periodic solutions (154)	0.78%

Table 1 Summary of the detected periodic attractors

The attractors we found are of different nature, from periodic, as the one presented in Fig. 4 to chaotic, with the example shown in Fig. 4 (c). The time series presented in Figure 4(d) suggests a presence of so called tumbling chaos [10],

the situation in which the pendulum rotates unpredictably in both directions and occasionally oscillates around the equilibrium position.
Table 1 shows a summary of the detected periodic attractors that constitute 90.33% of all the solutions. The column Period tells how many periods of excitation the system needs before it returns to the departure point in the phase space. Next one contains a short description of the attractor, while the last one, Probability, reflects the frequency of occurrence of the given solution in our simulation. It does not sum up to 100% because the rest (9.67%) is aperiodic and therefore difficult to classify.

4. Conclusions

In the article we presented a sample based approach for identifying the attractors of the dynamical systems. With our method we were able to find 164 different periodic attractors in the system composed of the Duffing oscillator and a tuned mass absorber. The extended basin stability method, shown in this work, can be applied to the systems with any number of degrees of freedom. Moreover, it captures the influence of the variations of the system parameters on the dynamics of the system. Basing on the samples we can also estimate the probability of occurrence of each attractor in case of the multistable zones. All these features make the approach an interesting tool for the dynamical analysis.

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